Decomposition of Hypercubes

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Award of the Degree of

Master of Philosophy
in
Mathematics

by
Annet Roy
(Reg. No. 1640037)

Under the Guidance of
Joseph Varghese
Associate Professor

CHRIST UNIVERSITY
BENGALURU, INDIA
Declared as Deemed to be University under Section 3 of UGC Act 1956

Department of Mathematics
CHRIST UNIVERSITY
BENGALURU, INDIA

November 2017

Approval of Dissertation

Dissertation entitled **Decomposition of Hypercubes** by Annet Roy, Reg. No. 1640037 is approved for the award of the degree of Master of Philosophy in Mathematics.

Examiner: ____________________ ____________________

Supervisor: ____________________ ____________________

Chairman: ____________________ ____________________

GRC: ____________________ ____________________

Date:
Place: Bengaluru

DECLARATION

I, Annet Roy, hereby declare that the dissertation, titled **Decomposition of Hypercubes** is a record of original research work undertaken by me for the award of the degree of Master of Philosophy in Mathematics. I have completed this study under the supervision of Dr Joseph Varghese, Associate Professor, Department of Mathematics.

I also declare that this dissertation has not been submitted for the award of any degree, diploma, associateship, fellowship or other title. I hereby confirm the originality of the work and that there is no plagiarism in any part of the dissertation.

Place: Bengaluru
Date:

Annet Roy
Reg. No. 1640037
Department of Mathematics
Christ University, Bengaluru.

CERTIFICATE

This is to certify that the dissertation submitted by Annet Roy, Reg. No. 1640037, titled **Decomposition of Hypercubes** is a record of research work done by her during the academic year 2016-2017 under my supervision in partial fulfillment for the award of Master of Philosophy in Mathematics.

This dissertation has not been submitted for the award of any degree, diploma, associateship, fellowship or other title. I hereby confirm the originality of the work and that there is no plagiarism in any part of the dissertation.

Place: Bengaluru
Date:

Dr Joseph Varghese
Associate Professor
Department of Mathematics
Christ University, Bengaluru.

Dr T V Joseph
Head of the Department
Department of Mathematics
Christ University, Bengaluru.

Acknowledgment

I would like to express my sincere gratitude to Dr Joseph Varghese for his profound guidance. The help and motivation given has inspired me to initiate my own research work. I am indebted to Dr T. V. Joseph the Head of the Department of Mathematics and Dr Mayamma Joseph coordinator of M.Phil programme for their support and encouragement. I also thank all the faculty members of Department of Mathematics, Christ University for their cordial support and valuable suggestions.

I express my gratitude to Christ University management in offering the essential infrastructural facilities, particularly the library. I would like to express my gratitude to my fellow friends for their valuable inputs. I thank Lord Almighty for His blessings and guidance. I also thank my parents, relatives for their constant support and encouragement.

Annet Roy

ABSTRACT

A hypercube Q_{2n} where n is a positive integer can be decomposed into 4^{n-1} copies of n-fan $F_{n,4}$ and a hypercube Q_{2n+1} can be decomposed into maximum 4^{n-1} copies of double n-fan $F^*_{n,4}$ and 3.4^{n-1} copies of K_2.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Background and History

Research in the field of graph theory has grown exponentially over the last few decades. Being rich in theoretical data as well as due to its apparent application to real life problems, graph theory as a branch, offers a wide range of inspiring and progressive ideas. A graph is a set of vertices representing objects and the connections between the objects are given by set of edges which are drawn between pairs of vertices. Graphs are used to model discrete structures which has the valuable feature of helping us to visualize, analyze or to generalize any issue which we may encounter.

The history of graph theory reverts to the year 1735 when the Swiss mathematician Leonhard Euler solved the Konigsberg bridge problem. The city of Konigsberg was located in Prussia (in Europe). The River Pregel flowed through the city dividing it into four land areas and seven bridges crossed the river at various locations. The Konigsberg bridge problem was an old puzzle to find whether there was any way to walk over all the bridges once and only once.

In 1852, it was observed that in a map of England, the countries could be coloured with four colours, in such a fashion that every two countries having a common boundary were coloured differently. This led to a much more general problem. That is, in a map consisting of various regions, can the regions be coloured with four or fewer colours, so that every two regions possessing a common boundary are coloured differently? The problems we have mentioned were not initially part of the graph theory as this theory had not been allocated then. This situation underwent a drastic change in 1891, when Julius Petersen, the Danish Mathematician, brought out the first purely theoretical article dealing with graphs as mathematical objects.

1.2 Outline of the Dissertation

This dissertation is titled as "Decomposition of Hypercubes" which is divided into four chapters. A study is made with respect to the family hypercube and how it can be decomposed into n-fan and double n-fan.

The first chapter contains all the basic definitions which helps in understanding the subsequent chapters. The chapter also has some examples which gives a better understanding of the graph theoretical concepts.

The second chapter is the review of literature. A detailed study is made regarding hypercubes in general, decomposition in hypercubes and certain results which aid in understanding the succeeding chapters are stated.

In the third chapter, an argument has been made regarding the decomposition of hypercubes into n-fan and double n-fan.

The fourth chapter presents the conclusion of the work done and also goes on to discuss the scope of future research in this area.

All the diagrams are generated with the help of Ti*k*Z.

1.3 Basic Terminology

The fundamental definitions and notations used throughout this dissertation are introduced in this subsection. These standard definitions are derived from the books written by Chartrand and Zhang [7] and West [25]. Through out this dissertation G denotes a simple, connected, finite and undirected graph.

Definition 1.3.1. A **graph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$, and a relation that associates with each edge having two vertices called the end points. A graph G

is notated as $G = (V,E)$. The number of vertices in a graph G is called the order of G and the number of edges in G is its size .

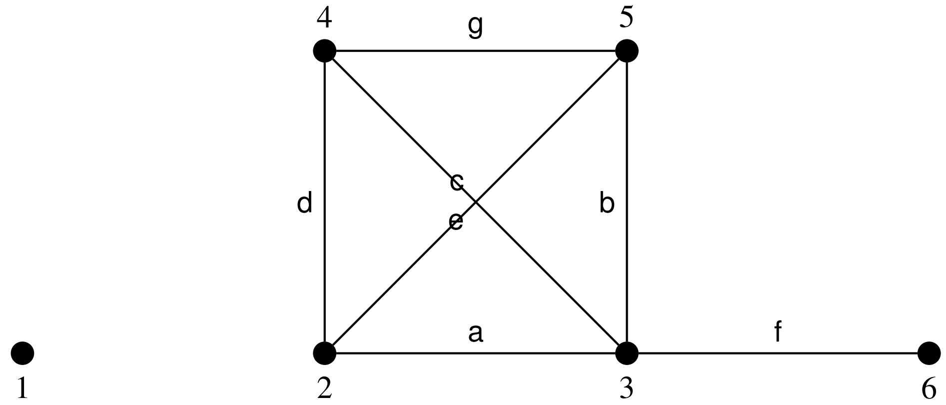


Figure 1.1: Graph G

Example 1.3.2. In the above figure 1.1, G is a graph of order 6 and size 7 with vertex set given by $V(G) = \{1,2,3,4,5,6\}$ and the edge set $E(G) = \{a,b,c,d,e,f,g\}$.

Definition 1.3.3. A graph with exactly one vertex is know as **trivial graph** and the graph with more than one vertex is know as **non-trivial graphs**.

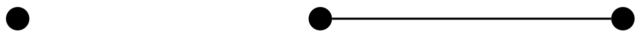


Figure 1.2: Trivial and Nontrivial graph

Definition 1.3.4. A **subgraph** of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We then write $H \subseteq G$ and say that **G contains H**

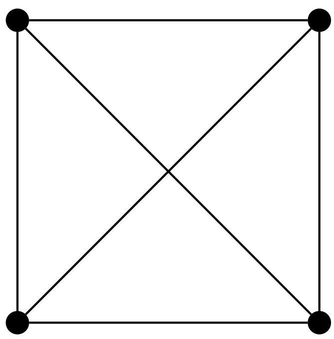


Figure 1.3: G_1

Figure 1.4: G_2

Figures G_1 and G_2 are the subgraphs of graph G in example 1.3.1.

Definition 1.3.5. A **spanning subgraph** H of a graph G is a subgraph with vertex set V(G).

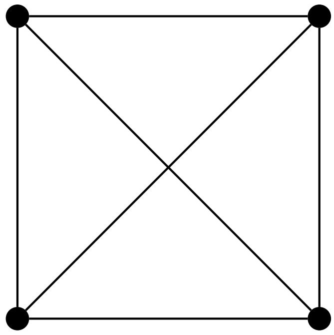


Figure 1.5: G

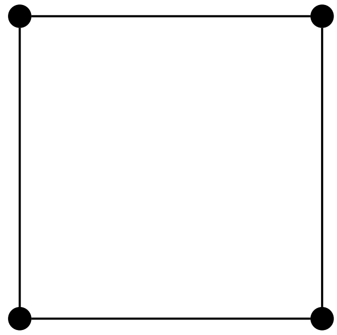


Figure 1.6: H

H is a spanning subgraph of G

Definition 1.3.6. A **loop** is an edge whose end points are equal. **Multiple edges** are edges with same pair of end points. A graph without any loops and multiple edges is called a **Simple graph**.

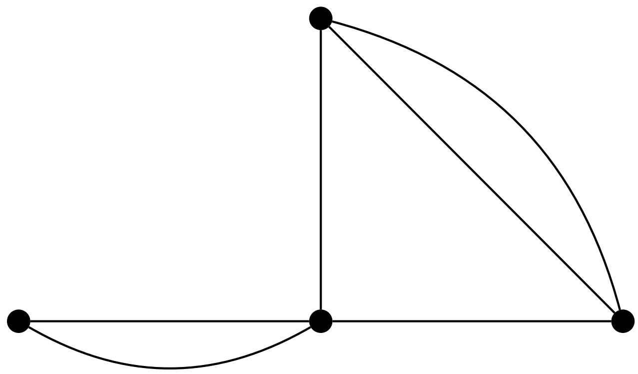


Figure 1.7: Graph with Multiple edges

Definition 1.3.7. If uv is an edge of G(V,E), then u and v are said to be **adjacent** in G. If there is no edge between the pair of vertices u and v, then u and v are **non-adjacent** vertices.

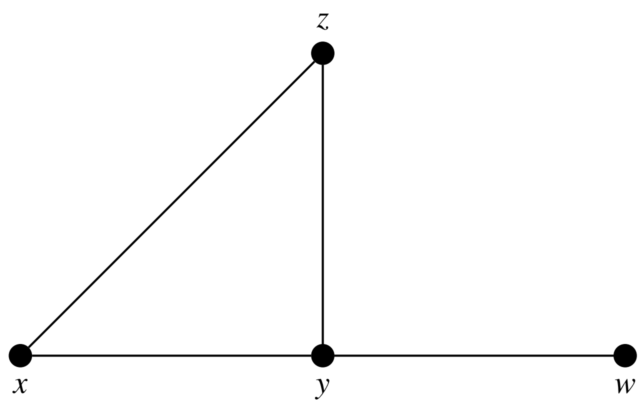


Figure 1.8: *x, y, z* are adjacent vertices

Definition 1.3.8. For any graph G, we define

$$\delta(G) = min\{d(v) : v \in V(G)\}$$

and

$$\Delta(G) = max\{d(v) : v \in V(G)\}$$

$\delta(G)$ is called the **minimum degree** and $\Delta(G)$ is called the **maximum degree** of G.
A graph in which $\delta(G)$ = $\Delta(G)$ is called a **regular graph** [25]. A graph is *k*-**regular** if the degree of all vertices of G is *k* [25].

Definition 1.3.9. A u-v **walk W** in a graph G is a sequence of vertices in G, beginning with u and ending with v such that consecutive vertices in the sequence are adjacent. If u = v, then the walk W is **closed** and if u $\neq$ v, then W is **open**. The number of edges encountered in the walk, including multiple occurrences of an edge, is called the **length** of the walk.

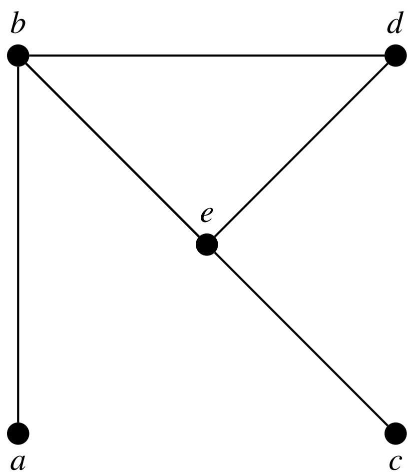


Figure 1.9: A walk

In Figure 1.9, W: a, b, e, c, e, d, b is a walk of length 6.

Definition 1.3.10. A u-v **trail** T in a graph G is defined to be a u-v walk in which no edge is traversed more than once.

In Figure 1.9, T: b, e, d, b is a trail.

Definition 1.3.11. A **Path** is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. P_k is a path on k vertices.

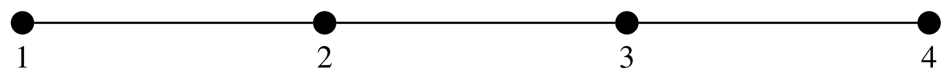


Figure 1.10: A path P_4

Definition 1.3.12. A **Cycle** is a graph with equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A cycle of odd length is called an **odd cycle** and a cycle of even length is called an **even cycle**. Cycle of length n is denoted as C_n.

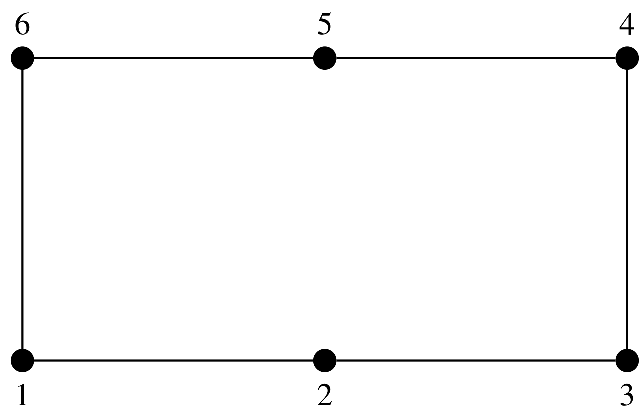


Figure 1.11: A cycle C_6

Definition 1.3.13. A graph is **Eulerian** if it has a closed trail containing all the edges. An **Eulerian circuit** in a graph is a circuit containing all the edges. A **Hamiltonian graph** is a graph with a spanning cycle. The cycle is called the **Hamiltonian cycle**.

Definition 1.3.14. An **Isomorphism** from a simple graph G to a simple graph H is a bijection $f: V(G) \longmapsto V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is isomorphic to H", written $G \cong H$, if there is a isomorphism from G to H.

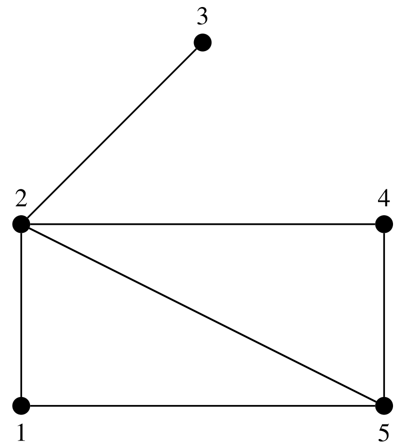


Figure 1.12: G_1

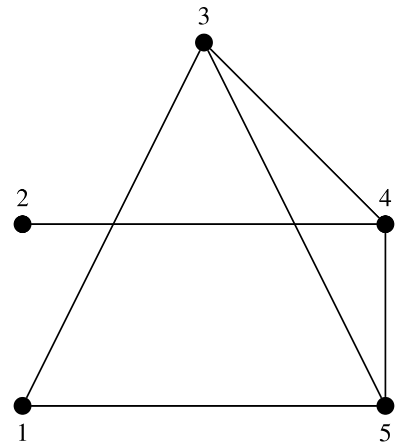


Figure 1.13: G_2

Graphs given in figures 1.12 and 1.13 and isomorphic to each other

Definition 1.3.15. The **complement** $\overline{G}$ of a simple graph G with vertex set V(G) defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. An **independent set** in a graph is a set of pairwise non-adjacent vertices and a **clique** is a set of pairwise adjacent vertices. The graph is **self complementary** if it is isomorphic to its complement.

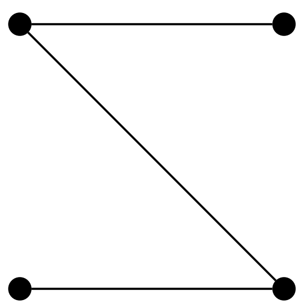


Figure 1.14: G

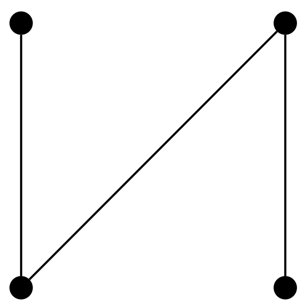


Figure 1.15: $\overline{G}$

Definition 1.3.16. A graph G is **complete** if every two distinct vertices of G are adjacent. A complete graph on n vertices is denoted by K_n.

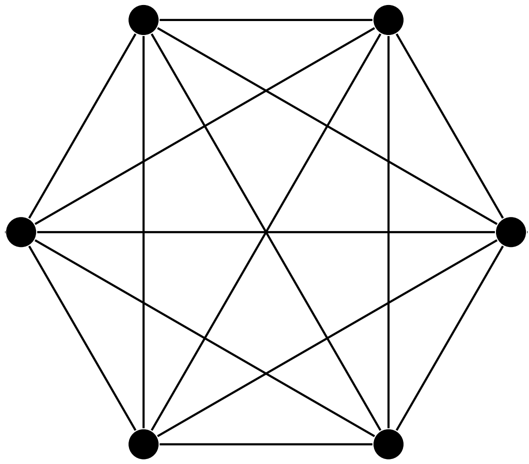


Figure 1.16: Complete graph K_6

Definition 1.3.17. A graph G is a **bipartite graph** if V(G) can be partitioned into two subsets U and W called the partite sets, such that every edge of G joins a vertex of U and a vertex of W.

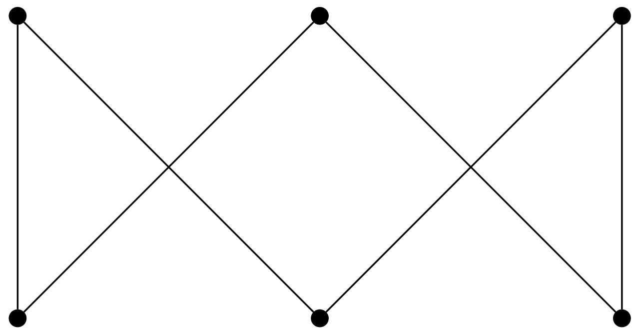


Figure 1.17: A bipartite graph

Definition 1.3.18. A graph G is **connected** if each pair of vertices in G belongs to a path; otherwise, G is disconnected. The **components** of a graph G are its maximal connected sub graphs.

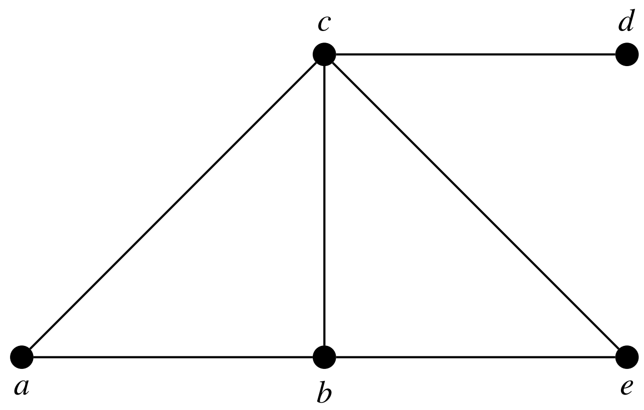


Figure 1.18: A connected graph

Definition 1.3.19. A **cut edge** or a **cut vertex** of a graph is an edge or vertex whose deletion increases the number of of components.

Definition 1.3.20. The **connectivity** of G, written $\kappa(G)$, is the minimum size of a vertex set S such that G-S is disconnected or has only one vertex. A graph G is **k-connected** if its connectivity is at least k.
The **edge connectivity** of G, written $\lambda(G)$, is the minimum size of the set F $\subset$ E(G) such that G-F has more than one component.

Definition 1.3.21. The **degree sequence** of a graph is the list of vertex degrees, usually written in the non-increasing order, as $d_1 \geq d_2 ... \geq d_n$.

Definition 1.3.22. A **directed graph** or **digraph** G is a triple consisting of a vertex set V(G), an edge set E(G), and a function assigning each edge an ordered pair of vertices. The first vertex of the ordered pair is the **tail** of the edge, and the second is the **head**; together, they are the **endpoints**.

Definition 1.3.23. An **orientation** of a graph G is a digraph D obtained from G by choosing an orientation (x $\rightarrow$ y or y $\rightarrow$ x) for each edge $xy \in$ E(G). An oriented graph is an orientation of a simple graph . A **tournament** is an orientation of a complete graph.

Definition 1.3.24. A graph with no cycle is **acyclic**. A **forest** is an acyclic graph. A **tree** is connected acyclic graph. A **leaf** or **pendant vertex** is a vertex of degree 1.

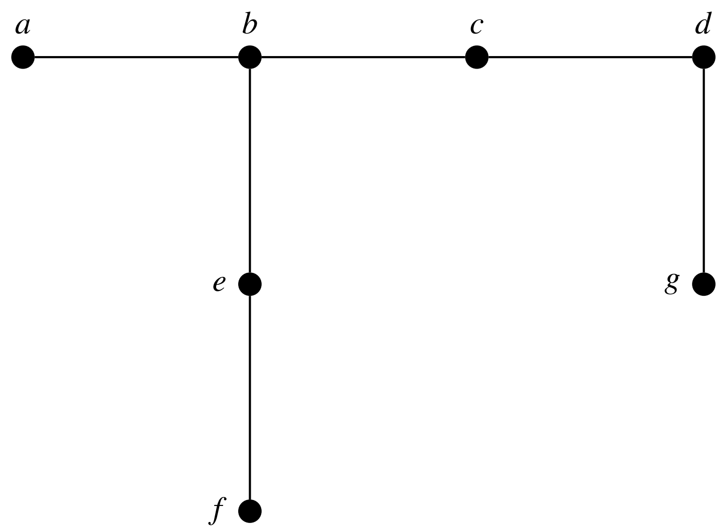


Figure 1.19: A tree

Definition 1.3.25. For any two points *u, v* of a graph we define the **distance** between *u* and *v* denoted by $d(u,v)$ is the length of the shortest $u - v$ path if such a path exists and infinite

otherwise.
The **diameter** of G, written *diam G* is the $max\{d(u,v) : u, v \in V\}$.
The **eccentricity** of a vertex u, denoted by *e(u)* is $max\{d(u,v) : v \in V\}$.
The **radius** of a graph G, written *rad G* is $min\{e(u) : u \in V\}$.
The **centre** of a graph G is the subgraph induced by the vertices of minimum eccentricity.

Definition 1.3.26. A **matching**, M in a graph G is a set of non loop edges with no shared end points. The vertices incident to the edges of a matching M are saturated by M. A matching that saturates every vertex is called a **perfect matching**.

Definition 1.3.27. Let G and H be two graphs with V(G) =$\{u_1, u_2, \cdots, u_m\}$ and V(H) = $\{v_1, v_2, \cdots, v_n\}$. Their **Cartesian Product** $G \square H$ has vertex set $V(G \square H) = V(G) \times V(H)$ and a vertex (u_1, v_1) is adjacent to (u_2, v_2) if and only if either $u_1 = u_2$ and v_1 adjacent to v_2 or $v_1 = v_2$ and u_1 adjacent to u_2.

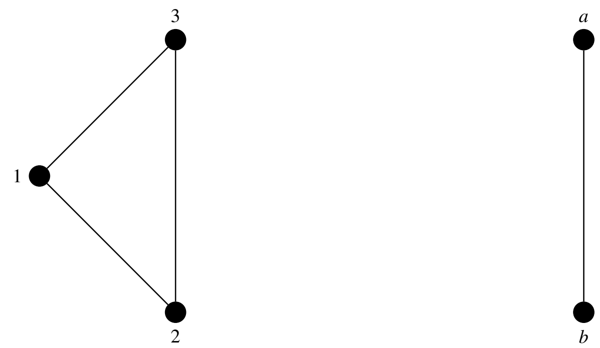


Figure 1.20: K_3 and K_2

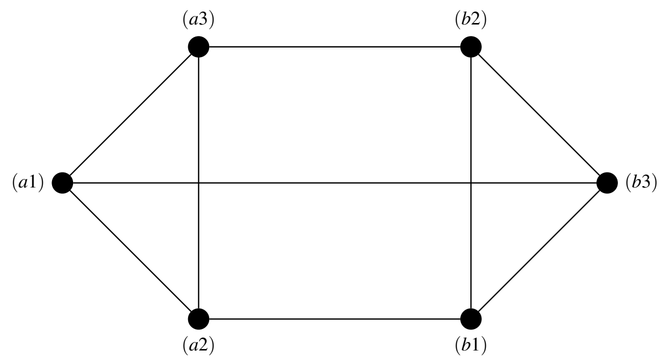


Figure 1.21: $K_3 \square K_2$

Definition 1.3.28. A **decomposition** of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

Decomposition of a graph G can be also said as, it is a set of subgraphs $H_1,...,H_k$ that partition the edges of G, that is, for all i and j , $\bigcup_{1\leq i\leq k} H_i = G$ and $E(H_i)\cap E(H_j) = \phi$.

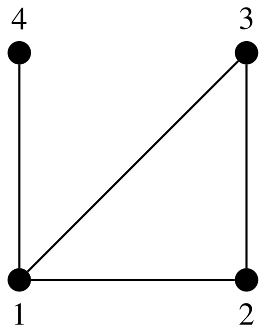


Figure 1.22: Graph G

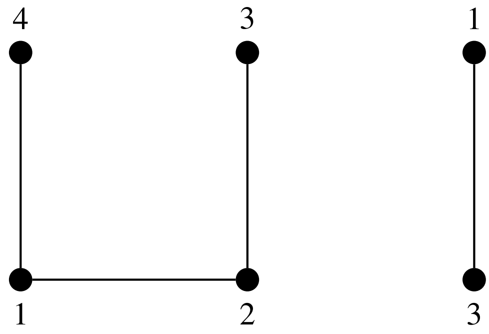


Figure 1.23: H_1 and H_2

One of the possible decomposition of graph G is shown in figure 1.23.

Chapter 2

Literature review

2.1 Hypercubes

Hypercubes are fundamental structures used in communication as well as in coding theory. In this section we define hypercubes and discuss the properties this family exhibits .

Definition 2.1.1. For any positive integer n, a n dimensional hypercube Q_n is a graph with the vertex set

$$V(Q_n) = \{X = x_1x_2...x_n : x_i \in \{0,1\}, 1 \leq i \leq n\}$$

and edge set E(Q_n) $=\{(X,Y)$: X = $x_1x_2...x_n$, Y = $y_1y_2...y_n$ and $x_i \neq y_i$ for exactly one i, 1$\leq$ i $\leq$ n [8]. We can also state this definition as, a hypercube Q_n is the simple graph whose vertices are the n-tuples with entries in $\{0,1\}$ and whose edges are the pairs of n-tuples that differ in exactly one position [25].

In a hypercube Q_n each vertex is a binary string denoted as X = $x_1x_2...x_n$ with n bits x_i = 0 or 1, therefore a hypercube is also known as a binary cube. A hypercube is also known in few other names such as n-cube, cube-connected network, cosmic cube and Boolean n-cube. A hypercube Q_n can also be defined in terms of product, it is the cartesian product of n copies of K_ns denoted by Q_n = $\underbrace{K_n \square K_n \square .....\square K_n}_{ntimes}$.

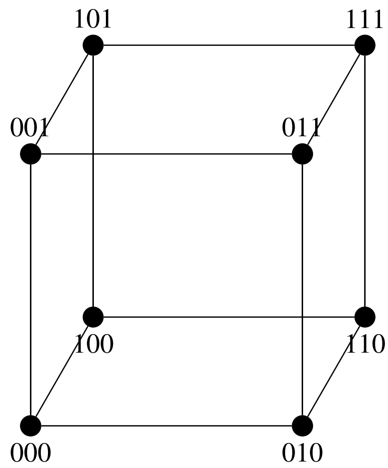


Figure 2.1: Hypercube Q_3

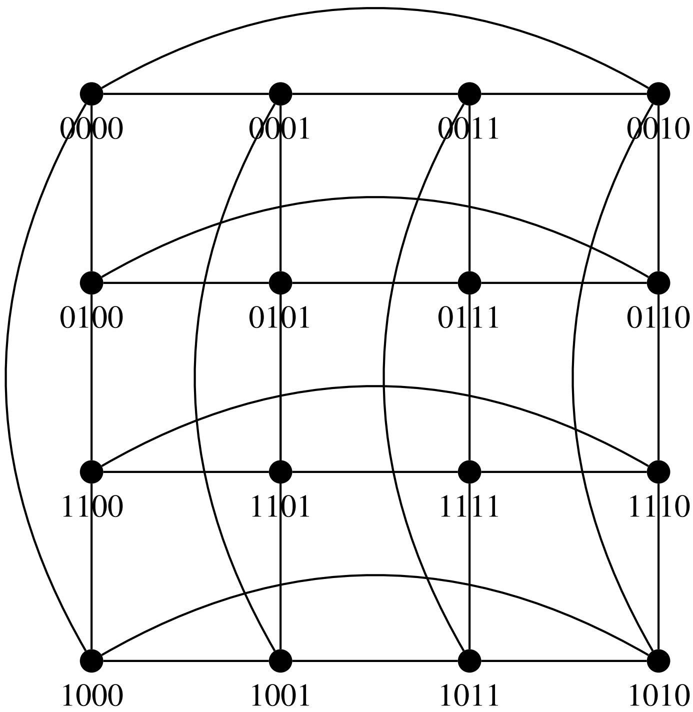


Figure 2.2: Q_4 as $\mathrm{K}_2\square K_2\square K_2\square K_2$

Properties of Hypercubes

- In a n dimension hypercube Q_n there are 2^n vertices.
- Q_n is a n regular graph.
- The total number of edges in Q_n is $n2^{n-1}$.

- The graph Q_n is connected as well as bipartite.
- For 1$\leq$ k $\leq$n-1, hypercube $Q_n \cong Q_k \Box Q_{n-k}$ in particularly $Q_n \cong K_2 \Box Q_{n-1}$.
- The vertex connectivity of Q_n is n.

There is a perfect matching of 2^{n-1} edges in Q_n. The deletion of these edges generates a graph with two components, each of which is isomorphic to Q_{n-1}. This operation of splitting is called a canonical decomposition of Q_n and is denoted by $Q_n = Q^0_{n-1} \Box Q^1_{n-1}$. The recursive construction as well as the splitting properties are often used in proving the structural properties of Q_n and this is done by induction on n.

Definition 2.1.2. Let U, V be binary strings of equal length. The Hamming distance H(U, V) between U and V is the number of positions at which U and V differ [8].

Definition 2.1.3. In a graph G, two *(x, y)*-paths are said to be parallel if they have no common internal vertices [8].

Proposition 2.1.4. *The following distance properties hold for a hypercube Q_n.*

1. *For any X, Y $\in$ V (Q_n), d(X, Y ;Q_n) = H(X, Y),*
2. *diam(Q_n) = n.*

For any two vertices X, Y belonging to the vertex set of the hypercube Q_n, the distance between the vertices d(X, Y ;Q_n) is the humming distance H(X, Y) which is less than or equal to n. Therefore, diam(Q_n) $\leq$ n [8]. Since d(X,$\overline{X}$;Q_n) = H(X,$\overline{X}$) = n, we can deduce that diam(Q_n) is n.

Proposition 2.1.5. *In Q_n, any two adjacent edges belong to exactly one cycle of length four.*

In a multiprocessor computing system, there is interchange of input happening amidst quite a few of its processors. The transmission of information occurs between processors through a series of intertwined processors. Apparently, the transmission would be faster if there is a subsequent increase in the number of 'alternative parallel paths' accessible for processing [8]. Correspondingly, the claim is that in the graph of an interconnection network, the distance between two given vertices should be small and there should be a subsequent increase in the number of parallel paths connecting any two vertices. Hypercube has these desirable properties and so can be used in such applications.

2.1.1 Characterization of hypercubes

In this section, we observed few properties of hypercubes. So, for a graph to be a hypercube all these properties are necessary conditions. Since many years there are few researchers trying to characterize hypercubes. They have succeeded in coming up with certain conditions which helps us to characterize the family of hypercubes. These necessary conditions and a few more additional conditions are sufficient for an arbitrary graph to be a hypercube. In this subsection, we present some of these characterizations.

Laborde and Hebbare [17] proved that a graph is an hypercube iff

1. G is connected (with δ(G) = n say),
2. every pair of adjacent edges lies in exactly one 4-cycle, and
3. $|V(G)| = 2^n$

Another important characterization in a Q_n is with respect to edge-coloring. In Q_n it can be easily noted that if we color an i^{th} dimensional edge with color i, we obtain a proper n-edge-coloring. Buratti [6] identified few more properties of this edge-coloring to obtain the following interesting characterization.

Theorem 2.1.6. *A graph G is a hypercube if and only if*

1. *G is connected,*
2. *G is regular (with degree of regularity, say, n),*
3. *G admits a proper n-edge-coloring satisfying the following conditions:*
   (a) *any two-colored path $\langle u,v,x,y,z\rangle$ of length four is closed; that is, it induces a cycle $\langle u,v,x,y,z=u\rangle$ of length four, and*
   (b) *any path whose edges have pairwise distinct colors is open.*

2.2 Decomposition of hypercubes

Decomposition problem dates back to the 19th century. Kirkman's problem of fifteen strolling school girls, Luca's dancing round problem, Dudney's problem of nine handcuffed prisoners and Euler's problem of thirty six army officers are the few popular problems of that time [5]. Early studies made upto 1985 regarding decomposition is comprehensively given by Juraj Bosak [5].

Research in graph decomposition was not only application oriented but had other theoretical studies made after 1985. Some of the theoretical studies include graphoidal decomposition [1], ascending subgraph decomposition [2] and continuous monotonic decomposition [9].

Lately, the n-dimensional hypercube graph Q_n has undergone extensive study. There is a lot of research studies being done with respect to hypercubes due to its wide range of application in the present era. Fink proved that 2^{n-1} isomorphic copies of edge disjoint trees on n edges can be obtained from the decomposition of an n dimension hypercube Q_n. He also observed that it was impossible to decompose the edge set of Q_n into spanning trees [10].

Hypercube is one of the effective network discovered for parallel computation. This network can efficiently simulate any other network of the same size. As a consequence of this most of the algorithms can be implemented directly on a hypercube without degrading the performance of the algorithm. This propelled Leighton to study hypercubes essentially because it was useful as the framework for distributed 'parallel processing supercomputers' [15].

Definition 2.2.1. A graph H decomposes into a graph G if one can write H as an edge-disjoint union of graphs isomorphic to G. H decomposes into D, where D is a family of graphs, when H can be written as a union of graphs each isomorphic to some member of D (where very member of D is represented at least once) [11].

Horak, Siran and Wallis demonstrated that the d-dimensional cube Q_d has an edge decomposition by isomorphic copies of any graph G with n edges each of whose blocks is either an even cycle or an edge. Furthermore, Q_n decomposes into D, where D is any set of six trees of size d [11].

Ramras independently proved that Q_n could be decomposed into 2^{n-1} isomorphic copies of any tree on n edges[17]. He also established that for an odd dimension hypercube Q_n, P_k which is a path of length k, yield an edge decomposition of Q_n[17].

Definition 2.2.2. A subgraph F of G is called a factor of G if it contains all the vertices of G. If in addition each component of F is isomorphic to H, we call F an H-factor of G. A decomposition of G into H-factors is called an H-factorization of G, and in this case we write H||G [10].

El-Zanati and Eynden[10] proved that, a hypercube Q_n has a C_s factorization of Q_d, if and only if d is even and s is equal to 2^t for t varying between 2 and d. He also obtained some results regarding the 1-factors in an odd dimension hypercube.

Song applies a totally distinct view to the hypercube of even dimension. He worked on the Hamiltonian decomposition of hypercube. He first obtained cycles of smaller length then used the merge operation to obtained the desired Hamilton cycles[20].

Studies have been conducted on maximum size families of edge disjoint spanning trees and their importance in solving the communication problems related to broadcasting. Barden put forward a method for deriving the maximum number of edge-disjoint spanning trees in a hypercube. The result would have applications to multicast communication in wormhole-routed parallel computers[4]. Chyun Ku, Wang and Hung also worked for results on more general product networks[14].

Decomposing the edge set of Q_n into spanning trees cannot be acheived. However for n being even, a direct method of n/2 edge-disjoint spanning trees in Q_n has been given by Roskind and Tarjan[19]. These trees are not isomorphic since there are n/2 remaining edges, which results in a path. For n being odd, a clear procedure has not been established yet[21].

The next question was about decomposition into isomorphic trees. Interestingly, for every tree T with n edges, the edge set of Q_n can be covered by 2^{n-1} trees isomorphic to T [10]. Wagner and Wild established that from a hypercube Q_n we can obtain n copies of a specific tree on 2^{n-1} edges[21].

The hypercube Q_{2k} may be decomposed into k edge-disjoint spanning trees and a matching of size k was proved by Karisani and Mahmoodian [12]. They also showed that a hypercube Q_{2k+1} may be decomposed into k edge-disjoint spanning trees with the remaining edges forming a forest with k components.

Kobeissia and Mollard demonstrated that double star like trees are embeddable into hypercubes [13]. It was done in three stages. They initially showed that a hypercube can be partitioned into vertex-disjoint cycles of even length. Secondly, defining a new family of graphs named MD-graphs, they showed that a hypercube can be decomposed into edge disjoint MD-graphs. In the final stage the decomposition of MD-graphs into double star like trees were obtained and this is in effect the same as decomposing hypercubes into double star like trees.

Definition 2.2.3. If H is isomorphic to a subgraph of G, we say that H divides G if there exist embeddings θ_1, θ_2, ..., θ_k of H such that $E(\theta_1(H))$, $E(\theta_2(H))$, ..., $E(\theta_k(H)$ is a partition of $E(G)$ [19]. For purposes of simplification we will often omit the embeddings, saying that we have an edge decomposition by copies of $E(H)$.

Mollard and Ramras perceived that a hypercube Q_n can be edge decomposed into copies of P_4,the path of length 4, for all n $\geq$ 5 [19]. Some of the other important results proved in this paper are

- For k $\geq$ 3, P_{2^k} does not divide Q_{2k+1}.
- Let H, G, $G^{'}$ be graphs. If H divides G and H divides $G^{'}$ then H divides G $\square$ $G^{'}$
- Suppose that the subgraph H of G edge-divides G. If G is n-regular and H is k-regular, then k divides n.

Some other researchers, Anick and Ramras made evident that a hypercube is edge decomposable into Hamiltonian cycles for Cartesian products of cycles[3]. They concentrated their work on the edge disjoint decompositions of hypercubes into paths of equal length. This subject matter has not attracted much attention in the past. Anick and Ramras also obtained a condition for n odd, $P_m <_D Q_n$ if and only if m $\leq$ n and m $\mid$ n.2^{n-1} where n$\leq$ 2^{32}. This statement can be proved in three parts. Initially, a case where the length of the path m is a power of 2. Secondly, edge decomposition of an odd dimension hypercube when m is a power of 2, for a small range of values of n just above m. And finally, the above result was extended from small range of n to 2^{32}.

Anick and Ramras showed that these two conditions are adequate for odd n $\leq$ 2^{32} and conjectured that this was true for all odd n. This conjecture was established by Erde [9]. He also proved for an even dimension hypercube Q_n edge disjoint Hamiltonian cycles decomposition is possible. Erde conjectured for n even and k such that $k|n.2^{n-1}$ and $k < 2^n$, then Q_n can be decomposed into P_k [11].

Chapter 3

Decomposition of Hypercube

3.1 Introduction

In the previous chapter, an evaluation has been made on how a hypercube decomposes into different families of graphs such as trees, paths and cycles. Decomposition of hypercubes are analyzed mainly due to its application in various fields such as coding theory and embedding systems. In this chapter, an effort has been taken to study the decomposition of hypercubes into graph classes namely n-fan and double n-fan.

3.2 Definitions

Definition 3.2.1. Let G = $(G_1, \ldots, G_k)$ be a family of graphs. A G-packing of a graph H is a set of pairwise vertex-disjoint subgraphs of H each of which is isomorphic to some graph in the family G [25]. A vertex of H is covered by the G-packing if it belongs to a subgraph of the G-packing.

Definition 3.2.2. An ***n*-fan** $F_{n,a}$ is a graph with n cycles of length a attached to a common vertex called the root vertex.

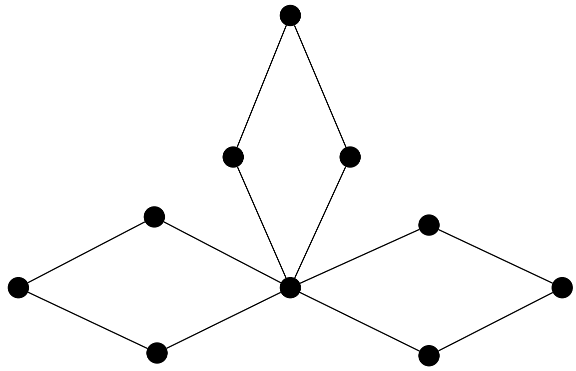


Figure 3.1: $F_{3,4}$

Definition 3.2.3. A **double *n*-fan** $F^*_{n,a}$ is a graph constructed by adding an edge between the two copies of an n-fan graph $F_{n,a}$ at its root vertex.

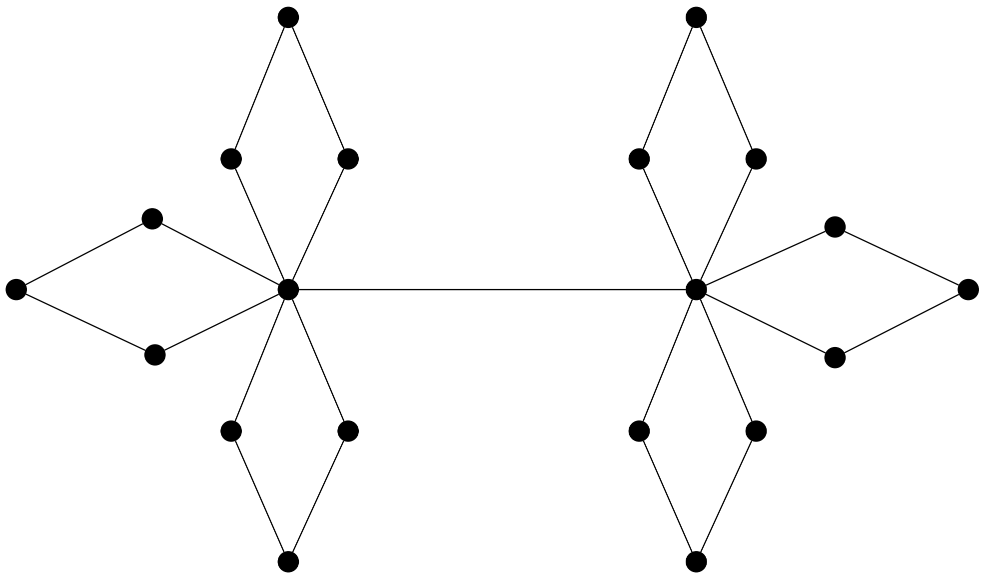


Figure 3.2: $F^*_{3,4}$

3.3 Even dimension Hypercubes

Theorem 3.3.1. *The Hypercube Q_{2n} can be decomposed into 4^{n-1} copies of $F_{n,4}$.*

Proof. The hypercube Q_{2n} has $n.2^{2n}$ edges. Each copy of an n-fan $F_{n,4}$ has $4n$ edges, since we have 4^{n-1} copies of $F_{n,4}$ the total number of edges would be $4^{n-1}.4n$ which is the total edges in a hypercube Q_{2n}.

In the case of a hypercube Q_{2n} where n is a positive integer, each vertex is a *2n*- tuple given by $(v_1 v_2 ... v_n ... v_{2n})$ where each $v_i \in \{0,1\}$. Moreover if $v_i = 0$ then $\overline{v_i} = 1$. Let the adjacent bits of the array of *2n* bits be paired to form *n* blocks represented as $(b_1 b_2 ... b_n)$ where each block $b_j = v_{2j-1} v_{2j}$ for $j \in \{1,2,...n\}$. To have the decomposition we need to identify

1. 4^{n-1} root vertices and
2. the n-cycles of length 4 attached to each root vertex.

The labels of the root vertices are generated in a table having n columns corresponding to n blocks and 4^{n-1} rows. The entries in one complete row will give the label of each root vertex. Let $b_{i,j}$ denote the entry corresponding to the i^{th} block and the j^{th} row in the table (Table 1). The entry $b_{i,j}$ is operated with an operator x denoted as $b^x_{i,j}$ where x takes vales 1, 2, 3 and 4. The action of the operator x is as follows

$$\left.\begin{aligned} b^1_{i,j} &= v_{2i-1} \ \ v_{2i} \\ b^2_{i,j} &= \overline{v_{2i-1}} \ \ v_{2i} \\ b^3_{i,j} &= \overline{v_{2i-1}} \ \ \overline{v_{2i}} \\ b^4_{i,j} &= v_{2i-1} \ \ \overline{v_{2i}} \end{aligned}\right\} \tag{3.1}$$

where $i \in \{1,2,...n\}$ and $j \in \left\{1,2,...4^{n-1}\right\}$

Generation of 4^{n-1} root vertices

Choose any vertex from the hypercube Q_{2n} say $(b_1 \ b_2 \ ... \ b_n)$ where each b_i is a block of two bits. Without laws of generality we can relabel this vertex as $b^1_{1,1} \ b^1_{2,1} \ ... \ b^1_{n,1}$. Obviously this is the first entry in the table (Table 1) corresponding to the first row. Note that the valve of the operator x is 1 in the first row.

In the column corresponding to block b_1 as we have 4^{n-1} rows and the operator has four possible values we partition the rows into 4 parts namely, 1 upto 4^{n-2}, $4^{n-2}+1$ upto 2.4^{n-2}, $2.4^{n-2}+1$ upto 3.4^{n-2}, $3.4^{n-2}+1$ upto $4.4^{n-2} = 4^{n-1}$. In general for a block b_i corresponding to the i^{th} column for $i \in \{1,2,...,n-1\}$ we partition the total rows into 4^i equal parts. Each part is considered to be an interval. In the case of any block b_i in the table the operator x changes from

$1 \to 2 \to 3 \to 4 \to 1$ an so on. The operator x changes after each interval and continues to be the same in the current interval. This process terminates once all the rows are filled.

To fill the entries $b^x_{n,j}$ corresponding to the n^{th} column for $j \in \{1,2,...4^{n-1}\}$ we find an a such that if $j \equiv a \pmod{16}$ and

- $a \in \{1,8,11,14\}$ then $x = 1$.
- $a \in \{2,5,12,15\}$ then $x = 2$.
- $a \in \{3,6,9,0\}$ then $x = 3$.
- $a \in \{4,7,10,13\}$ then $x = 4$ where x is the operator.

However, we know that Q_{2n} is the cartesian product of n copies of C_4, that is, $Q_{2n} = C_4 \square C_4 \square \ldots . \square C_4$. This implies that each induced cycle, i.e., C_4 in the cartesian product corresponds to a block b_i of Q_{2n}. Every cycle C_4 has four possible labels. In the above mentioned method we observe that at each stage keeping the block b_i fixed we can use all the possible four labels for the block b_{i+1} for $i \in \{1,2,...,n-2\}$. Thus entries in the block b_n is made in such a way no two root vertices generated in the table could be adjacent. This way we get all the 4^{n-1} root vertices.

	b_1	b_2	b_3	.	.	b_{n-3}	b_{n-2}	b_{n-1}	b_n
1	$b^1_{1,1}$	$b^1_{2,1}$	$b^1_{3,1}$	.	.	$b^1_{n-3,1}$	$b^1_{n-2,1}$	$b^1_{n-1,1}$	$b^1_{n,1}$
2	$b^1_{1,2}$	.	.	.	.	$b^1_{n-3,2}$	$b^1_{n-2,2}$	$b^2_{n-1,2}$	$b^2_{n,2}$
3	$b^1_{1,3}$	.	.	.	.	$b^1_{n-3,3}$	$b^1_{n-2,3}$	$b^3_{n-1,3}$	$b^3_{n,3}$
4	$b^1_{1,4}$	.	.	.	.	$b^1_{n-3,4}$	$b^1_{n-2,4}$	$b^4_{n-1,4}$	$b^4_{n,4}$
5	$b^1_{1,5}$	.	.	.	.	$b^1_{n-3,5}$	$b^2_{n-2,5}$	$b^1_{n-1,5}$	$b^2_{n,5}$
6	$b^1_{1,6}$					$b^1_{n-3,6}$	$b^2_{n-2,6}$	$b^2_{n-1,6}$	$b^3_{n,6}$
7	$b^1_{1,7}$					$b^1_{n-3,7}$	$b^2_{n-2,7}$	$b^3_{n-1,7}$	$b^4_{n,7}$
8	$b^1_{1,8}$					$b^1_{n-3,8}$	$b^2_{n-2,8}$	$b^4_{n-1,8}$	$b^1_{n,8}$
9	$b^1_{1,9}$					$b^1_{n-3,9}$	$b^3_{n-2,9}$	$b^1_{n-1,9}$	$b^3_{n,9}$
10	.					$b^1_{n-3,10}$	$b^3_{n-2,10}$	$b^2_{n-1,10}$	$b^4_{n,10}$
11	.					$b^1_{n-3,11}$	$b^3_{n-2,11}$	$b^3_{n-1,11}$	$b^1_{n,11}$
12	.					$b^1_{n-3,12}$	$b^3_{n-2,12}$	$b^4_{n-1,12}$	$b^2_{n,12}$
13	.					$b^1_{n-3,13}$	$b^4_{n-2,13}$	$b^1_{n-1,13}$	$b^4_{n,13}$
14	.					$b^1_{n-3,14}$	$b^4_{n-2,14}$	$b^2_{n-1,14}$	$b^1_{n,14}$
15	.					$b^1_{n-3,15}$	$b^4_{n-2,15}$	$b^3_{n-1,15}$	$b^2_{n,15}$
16	.					$b^1_{n-3,16}$	$b^4_{n-2,16}$	$b^4_{n-1,16}$	$b^3_{n,16}$
.	.						.	.	.
.	.						.	.	.
.	.						.	.	.
4^{n-2}	$b^1_{1,4^{n-2}}$						.	.	.
.	.						.	.	.
.	.						.	.	.
.	.						.	.	.
2.4^{n-2}	$b^2_{1,2.4^{n-2}}$						.	.	.
.	.						.	.	.
.	.						.	.	.
.	.						.	.	.
3.4^{n-2}	$b^3_{1,3.4^{n-2}}$						.	.	.
.	.						.	.	.
.	.						.	.	.
.	.						.	.	.
4^{n-1}	$b^4_{1,4^{n-1}}$	$b^4_{2,4^{n-1}}$	.	.	.	.	$b^4_{n-2,4^{n-1}}$	$b^4_{n-1,4^{n-1}}$	$b^4_{n,4^{n-1}}$

Table 3.1: Generation of 4^{n-1} root vertices

Generation of the n C_4s attached to a given root vertex.

Step 1 Let (b_1 b_2 ... b_n) be a root vertex of a hypercube Q_{2n} . Let C_4^k represents a cycle of length four which is the k^{th} leaf attached to the root vertex where $k \in \{1,2,...n\}$. The vertices belonging to the cycle in the k^{th} leaf can be generated by varying the block $b_k \in$ (b_1 b_2 ... b_n) while keeping all the other *n-1* blocks fixed. The block b_k is operated with the operator x as defined in equation (1). The cycle thus obtained is given below (figure 3.3)

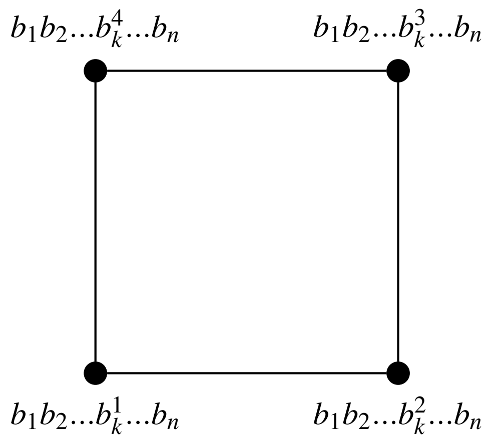


Figure 3.3: Generation of Cycle

Step 2 Repeat the process for all the values of $k \in \{1,2,...,n\}$ to generate all the n leaves for a single root vertex.

Step 3 Repeat step 1 and 2 with all other root vertices to get 4^{n-1} copies of $F_{n,4}$.

□

Example 3.3.2. A hypercube Q_4 can be decomposed into 4 copies of $F_{2,4}$.

A hypercube Q_4 has 16 vertices and 32 edges. Each vertex label is a 4-tuple, rewriting each of them to form blocks b_1 and b_2. To show the decomposition we need to identify the 4 root vertices as well as the two cycles attached to each root vertex.

Generation of the root vertices.
Choose any vertex of Q_4, say 0000 be the first root vertex.

	b_1	b_2
1	$b^1_{2,1}$	$b^1_{1,1}$
2	$b^2_{2,2}$	$b^2_{1,2}$
3	$b^3_{2,3}$	$b^3_{1,3}$
4	$b^4_{2,4}$	$b^4_{1,4}$

	b_1	b_2
1	00	00
2	01	01
3	11	11
4	10	10

Table 3.2: Generation of the root vertices

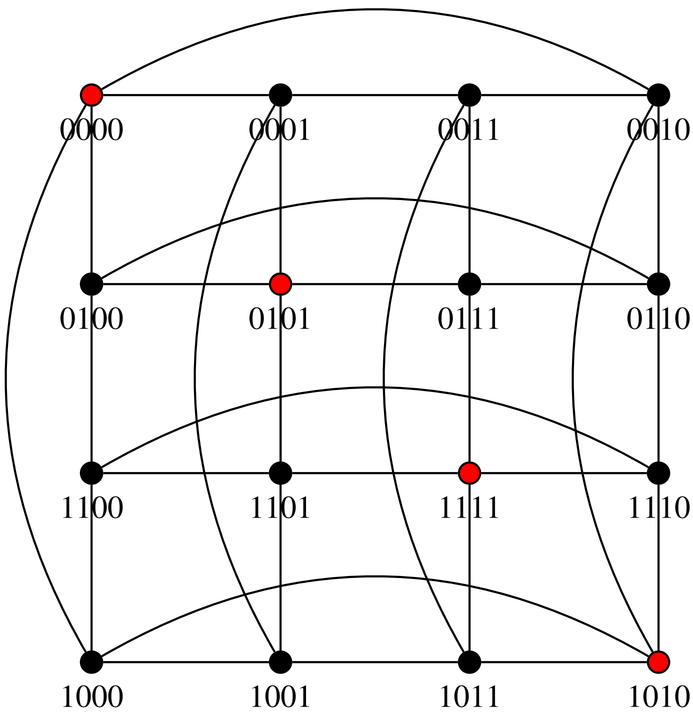
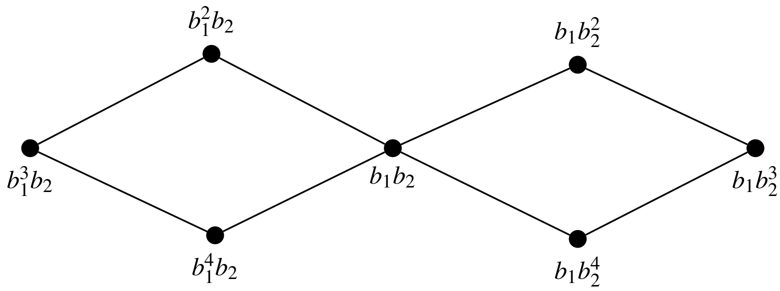


Figure 3.4: The root vertices of Q_4 are colored in red

Generation of the cycles

The figure below shows how to generate the cycles attached to a root vertex in case of Q_4.



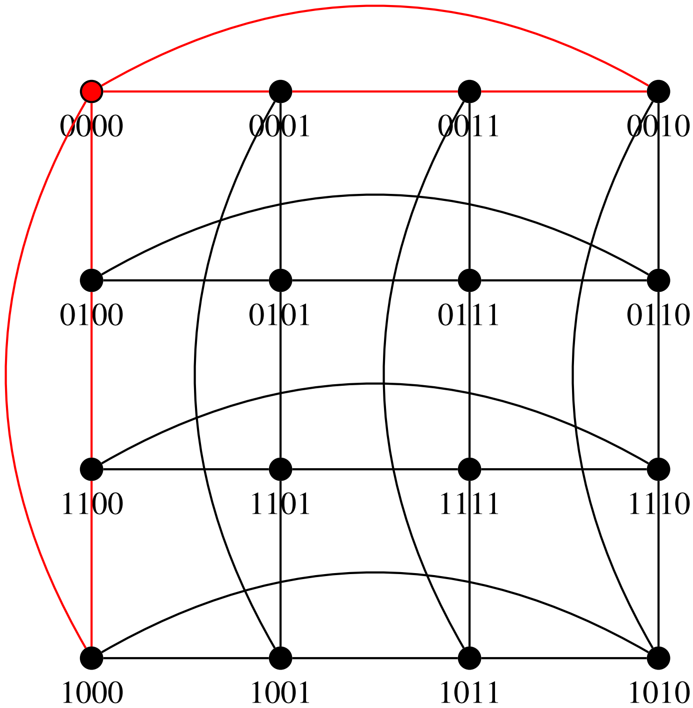


Figure 3.5: 1^{st} copy of $F_{2,4}$ in Q_4

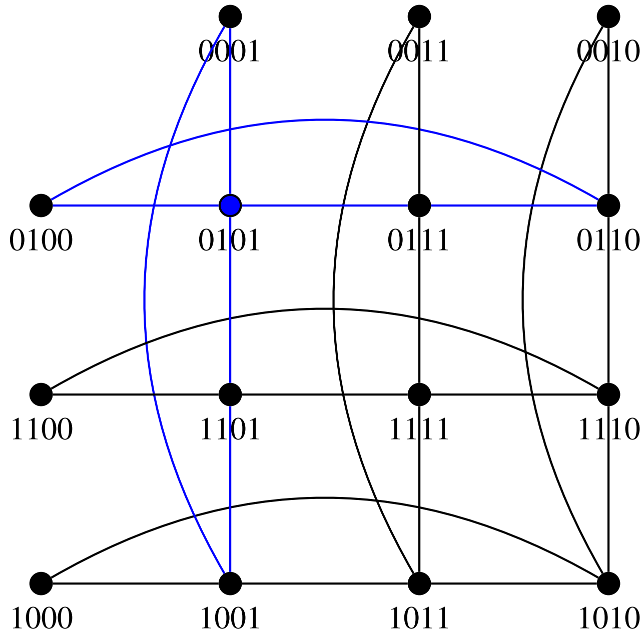


Figure 3.6: 2^{nd} copy of $F_{2,4}$ in Q_4

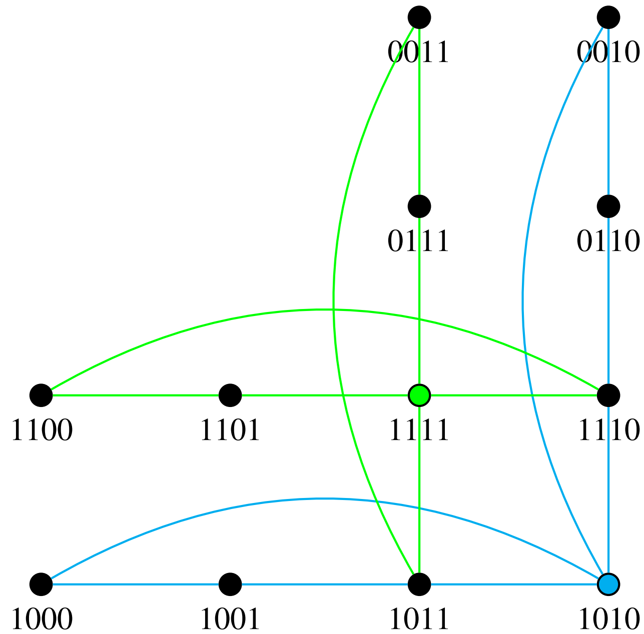


Figure 3.7: 3^{rd} and 4^{th} copies of $F_{2,4}$ in Q_4

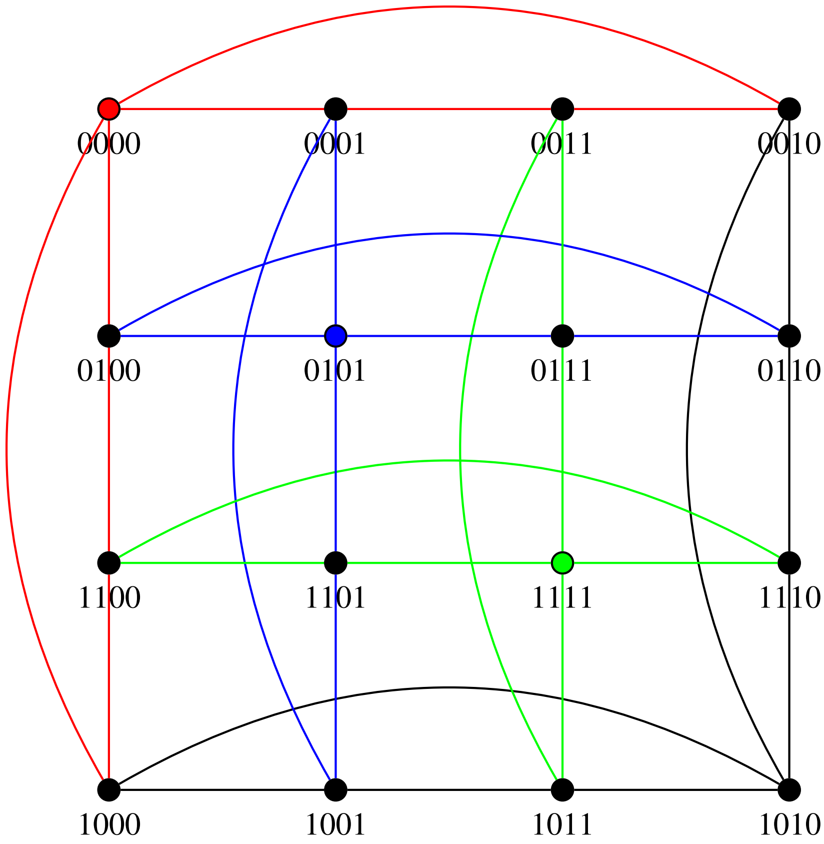


Figure 3.8: Q_4 decomposed into 4 copies of $F_{2,4}$

3.4 Odd dimension Hypercubes

Theorem 3.4.1. *The Hypercube Q_{2n+1} can be decomposed into 4^{n-1} copies of $F^*_{n,4}$ and 3.4^{n-1} copies of K_2.*

Proof. The hypercube Q_{2n+1} has $(2n+1).4^n$ edges. Each copy of a double n-fan $F^*_{n,4}$ has $8n+1$ edges. For 4^{n-1} copies of $F^*_{n,4}$ and 3.4^{n-1} copies of K_2 the total number of edges is $(4^{n-1})(8n+1)+3.4^{n-1} = (2n+1).4^n$. This is the total edges in a hypercube Q_{2n+1}.

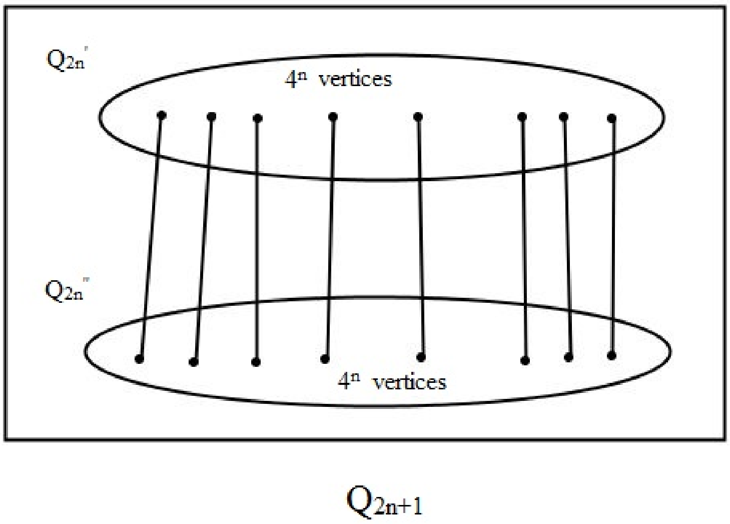


Figure 3.9: Q_{2n+1} is the cartesian product of Q_{2n} and K_2

The hypercube Q_{2n+1} is the cartesian product of Q_{2n} and K_2. Hence every Q_{2n+1} would have two copies of Q_{2n} and the edges between them. Let the two copies of Q_{2n} in Q_{2n+1} be $Q^{'}_{2n}$ and $Q^{''}_{2n}$. Each vertex of a hypercube Q_{2n+1} is a 2n+1 tuple. Let the vertices of $Q^{'}_{2n}$ and $Q^{''}_{2n}$ in Q_{2n+1} be labeled as $(v_1v_2...v_n...v_{2n}\ v_{2n+1})$ and $(v_1v_2...v_n...v_{2n}\ \overline{v_{2n+1}})$ respectively. These vertex labels for $Q^{'}_{2n}$ and $Q^{''}_{2n}$ can be rewritten as $(b_1b_2...b_n\ v_{2n+1})$ and $(b_1b_2...b_n\ \overline{v_{2n+1}})$ where each block $b_j = v_{2j-1}v_{2j}$ for $j \in \{1,2,...n\}$.

From the definition of a double n-fan we note that each copy requires two root vertices. These root vertices are carefully identified in such a way that one root vertex is chosen from the copy of $Q^{'}_{2n}$ in Q_{2n+1} and the other one is chosen from the copy of $Q^{''}_{2n}$ in Q_{2n+1}.

Consider the blocks $(b_1b_2...b_n)$ from the vertex label of Q_{2n+1}. We can generate the labels of 4^{n-1} vertices which serve as root vertices for $F_{n,4}$ by theorem 3.3.1. Once these vertices are generated, then we add a bit v_{2n+1} for all the 4^{n-1} labels and set it to 0. This would give the required root vertices in $Q^{'}_{2n}$. If the v^{th}_{2n+1} bit is set as 1 for all the 4^{n-1} vertex labels generated, then this would give the root vertices in $Q^{''}_{2n}$.

For the above labels of the root vertices generated there will be 2.4^{n-1} copies of $F_{n,4}$ in Q_{2n+1} by theorem 3.3.1. From the labeling pattern of the root vertices in $Q^{'}_{2n}$ and $Q^{''}_{2n}$ we find there are 4^{n-1} copies of $F^{*}_{n,4}$. This is because there are 4^{n-1} pairs of root vertices which can be identified in such a way that in each pair, one root vertex is chosen from the copy $Q^{'}_{2n}$ in Q_{2n+1} and the other is chosen from the copy $Q^{''}_{2n}$ in Q_{2n+1}. Theses root vertices are adjacent since their vertex labels differ exactly by one bit.

By the construction of Q_{2n+1} we know that $Q^{'}_{2n}$ would have exactly 4^n vertices adjacent to the same number of vertices of $Q^{''}_{2n}$. Since 4^{n-1} edges are already a part of 4^{n-1} copies of $F^*_{n,4}$ the remaining edges which are not a part of double n-fan decomposition is 3.4^{n-1}. Obviously this forms 3.4^{n-1} copies of K_2. Thus we get 4^{n-1} copies of double n-fan $F^*_{n,4}$ and 3.4^{n-1} copies of K_2.

The maximum number of copies of $F^*_{n,4}$ that can be obtained by the disjoint edge decomposition of a hypercube Q_{2n+1} is 4^{n-1}. To prove the above statement let us assume that, there exist atleast 4^{n-1}+1 copies obtained by the decomposition of Q_{2n+1}.

$F^*_{n,4}$ cannot be a subgraph of $Q^{'}_{2n}$ or $Q^{''}_{2n}$, that is, no copy of $F^*_{n,4}$ will lie completely in either $Q^{'}_{2n}$ or $Q^{''}_{2n}$. Since there exist no two root vertices whose degrees are 2n+1 and which are adjacent to each other. The only possibility is to have one of the root vertex of $F^*_{n,4}$ in $Q^{'}_{2n}$ and the other in $Q^{''}_{2n}$.

We assumed that the graph Q_{2n+1} has 4^{n-1}+1 copies of $F^*_{n,4}$. Therefore each copy of $F^*_{n,4}$ would have its one root vertex in $Q^{'}_{2n}$ and the other one in $Q^{''}_{2n}$. Thus there exist 4^{n-1}+1 root vertices in $Q^{'}_{2n}$ as well as in $Q^{''}_{2n}$. However the maximum number of non adjacent root vertices that can be there in the copy of $Q^{'}_{2n}$ or $Q^{''}_{2n}$ is 4^{n-1} by theorem 3.3.1. This shows that there is atleast a pair of root vertex which are adjacent in the copy of $Q^{'}_{2n}$ as well as in $Q^{''}_{2n}$. Obviously due to the adjacent pair of root vertices, there exist less than 4^{n-1}+1 copies of $F^*_{n,4}$ which is a contradiction to our assumption. Hence the maximum number of copies of $F^*_{n,4}$ that can be obtained by the edge decomposition of a hypercube Q_{2n+1} is 4^{n-1}.

□

Example 3.4.2. A hypercube Q_5 can be decomposed into 4 copies of $F^*_{2,4}$ and 12 copies of K_2.

In the example below we carefully identify the root vertices and the cycles attached to it. Each copy of $F^*_{2,4}$ is colored differently. The edges that are colored in black represent the K_2s. It is observed that, there is no more possible copy of $F^*_{2,4}$ that can be identified in Q_5.

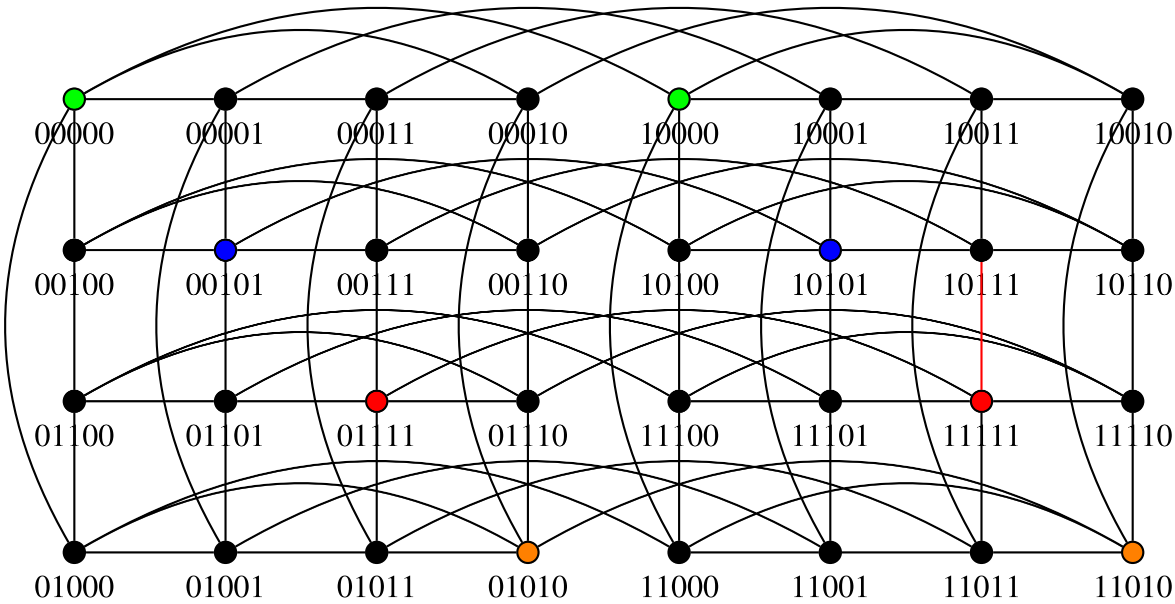


Figure 3.10: The root vertices of Q_5 are coloured in the figure

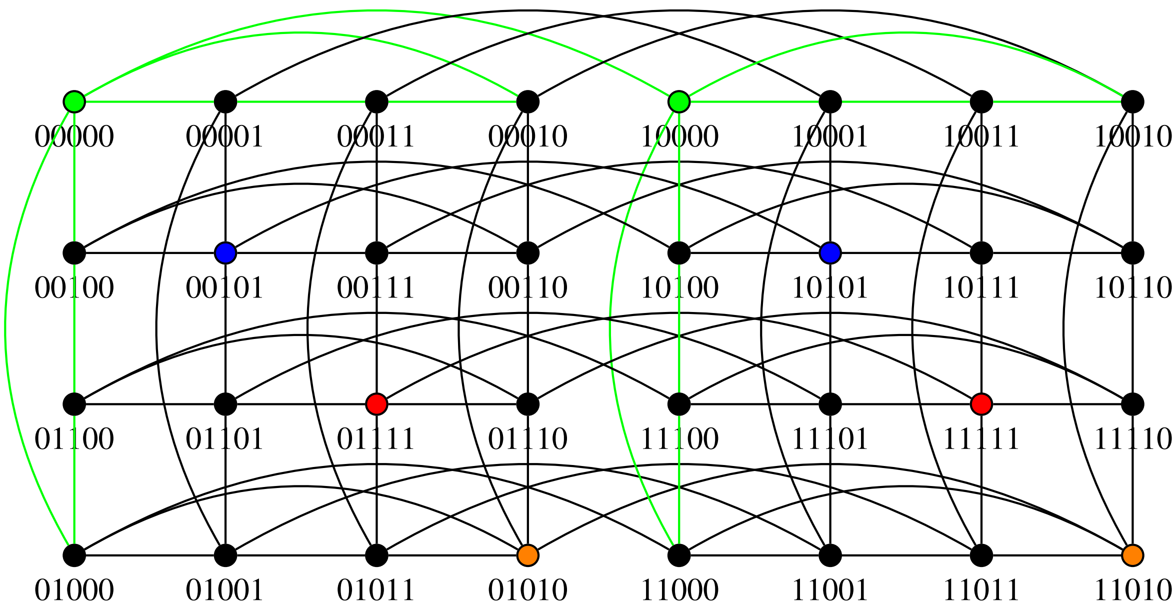


Figure 3.11: A copy of $F^*_{2,4}$ is coloured

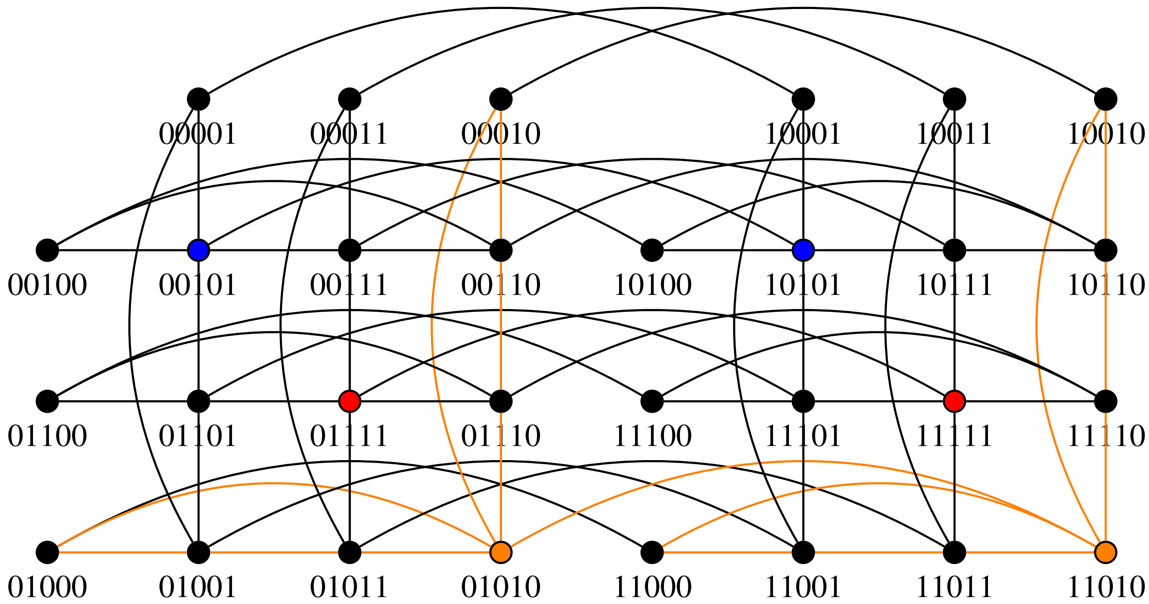


Figure 3.12: The 2^{nd} copy of $F^*_{2,4}$ is coloured

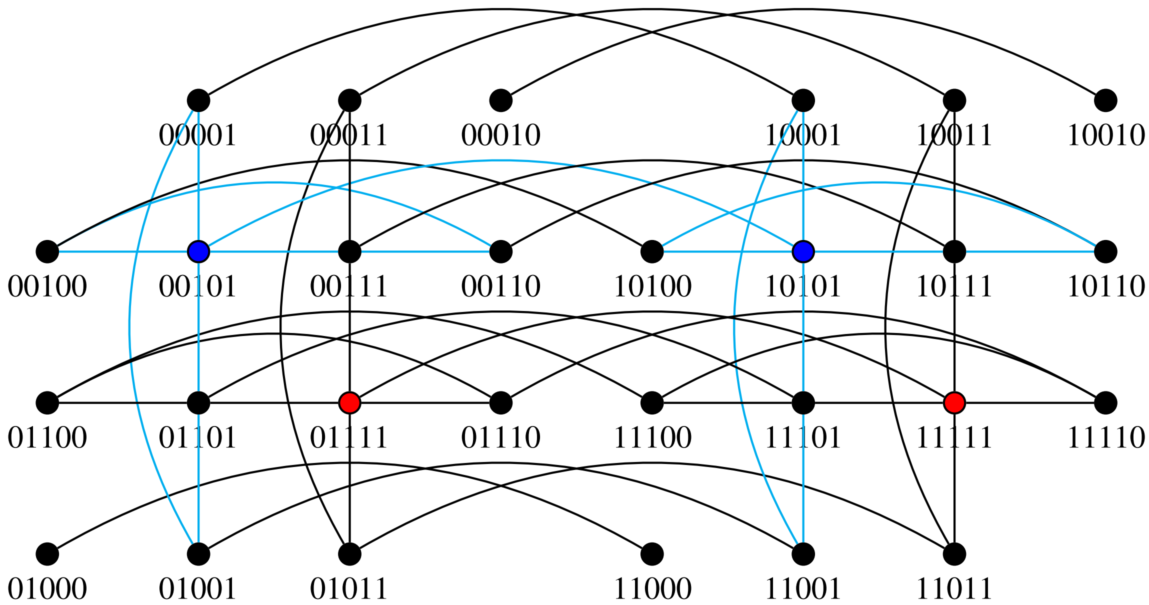


Figure 3.13: The 3^{rd} copy of $F^*_{2,4}$ is coloured

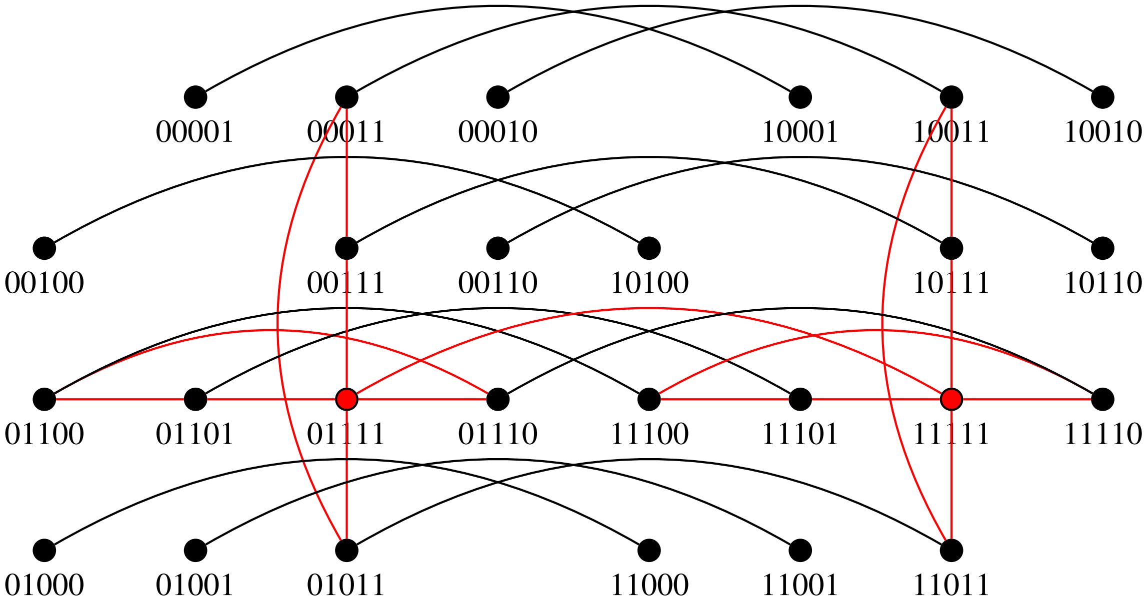


Figure 3.14: 4^{th} copy of $F^*_{2,4}$

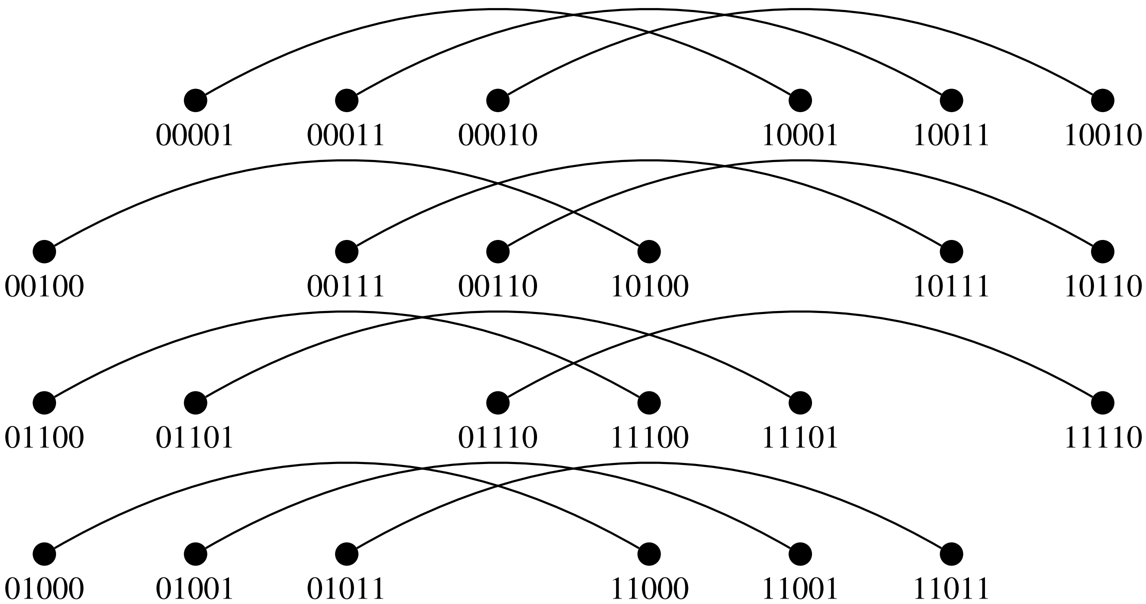


Figure 3.15: Remaining 12 edges form K_2s

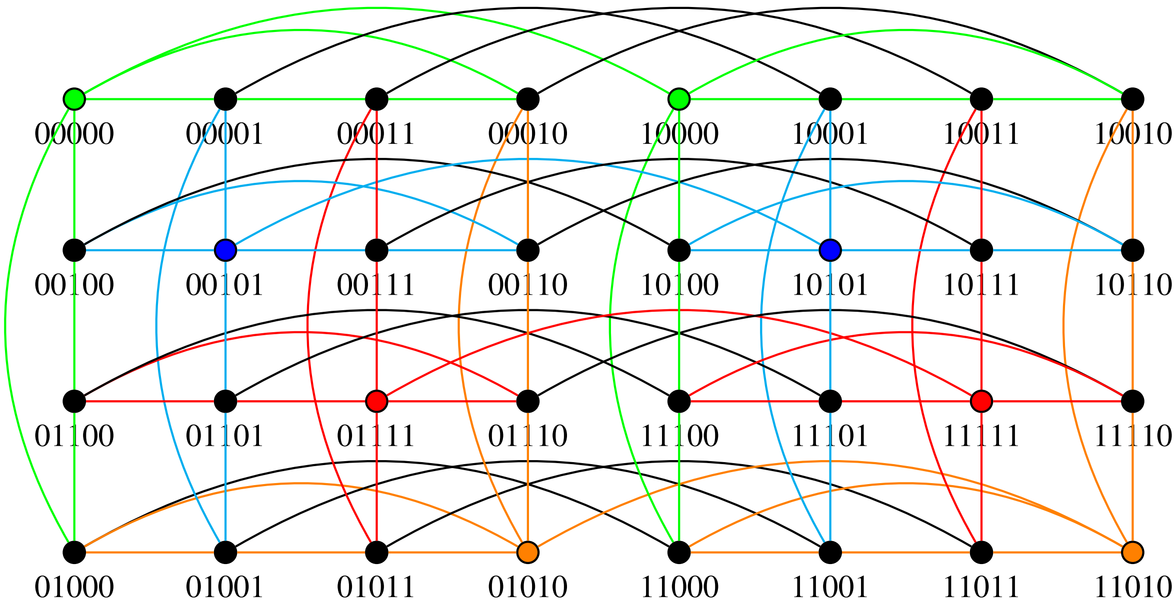


Figure 3.16: Q_5 is decomposed into 4 copies of $F^*_{2,4}$ and 12 copies of K_2

3.5 Conclusion

In this chapter we made a study on how the n dimension hypercube could be decomposed into n-fans and double n-fans. A systematic construction of how the root vertices as well as the n cycles attached to it generated is mentioned. A hypercube Q_{2n} where n is a positive integer can be decomposed into 4^{n-1} copies of n-fan $F_{n,4}$. A double n-fan $F^*_{n,4}$ packing into a hypercube Q_{2n+1} is established.

Chapter 4

Conclusion

Graphs play a very major role in any situation pertaining to either the physical, biological, social and other significant areas in life. For instance, in Computer Science, graphs are displayed to symbolize intercommunication, data coordination, estimation devices, the flow of computation and the like. To elucidate, the link structure of a website can be depicted by a directed graph, in which the nodes portray web pages and directed edges exhibit links from page to page. Likewise, we encounter a parallel approach while dealing with issues relating to travel, biology, computer chip design and many such correlated domains.

In our study, we have seen the decomposition of a hypercube into n-fans and double n-fans. As mentioned earlier, one can explore if continuous monotonic decomposition of hypercubes into n-fans and double n-fans is possible? If so, what are the possible n values in Q_n for which this kind of decomposition exists?

Consider Q_{10} (n = 5), we can verify the continuous monotonic decomposition of Q_{10} into 128 copies of $F_{1,4}$, $F_{2,4}$, $F_{3,4}$ and $F_{4,4}$ each.

Likewise, one can explore if a hypercube decomposes into odd or even ascending decomposition of n-fans or double n-fans. What are the possible values of n which have such a decomposition?

This study can be extended to different kinds of decompositions with respect to the graph classes namely n-fans and double n-fans. Since we have shown the decomposition, embedding these graph families n-fans and double n-fans into an hypercube, is also possible.

Bibliography

[1] B.D. Acharya and E. Sampathkumar, “Graphoidal cover and graphoidal covering number of a graph," *Indian J. Pure Appl. Math.*, vol. 18, no. 10, pp. 882-890, 1987.

[2] Y. Alavi *et al.*, “The ascending subgraph decomposition problem," *Congressum Numerantium*, vol. 58, pp. 1-7, 1987.

[3] D. Anick and M. Ramras, “Edge decompositions of hypercubes by paths," *Austral. J. Combin.*, vol. 61, no. 3, pp. 210-226, 2015.

[4] B. Barden *et al.*, “On edge-disjoint spanning trees in hypercubes," *Information Processing Letters*, vol. 70, no. 1, pp. 13-16, 1991.

[5] J. Bosak, *Decomposition of graphs*, Kluwer, London, 1990.

[6] M. Buratti , “Edge-colourings characterizing a class of cayley graphs and a new characterization of hypercubes," *Discrete Math.*, vol. 161, no. 1-3, pp. 291-295, 1996.

[7] G. Chartrand and P. Zhang, *Introduction to the Theory of Graphs*, New Delhi,India : Tata McGraw-Hill Education, 2012.

[8] S.A. Choudum, S. Lavanya and V. Sunitha, *Handbook of Graph Theory, Combinatorial Optimization, and Algorithms*, CRC Press, Taylor and Francis Group, 2015, pp.587-626.

[9] N. G. Dhas and J.J. Paulraj, “Continuous monotonic decomposition of graphs," *Int. J. of Manage. and Systems.*, vol. 16, no. 3, pp. 333-344, 2000.

[10] S.I. El-Zanati and C. V. Eynden, “Cycle factorizations of cycle products," *Discrete Math.*, vol. 189, no. 1 - 3, pp. 267-275, 1998.

[11] J. Erde, “Decomposing cubes into paths," *Discrete Math.*, vol. 336, pp. 41 - 45, 2014.

[12] J.F. Fink, “On the decomposition of n-cubes into isomorphic trees," *J. Graph Theory*, vol. 14, no. 4, pp. 405-411, 1990.

[13] P. Horak, J. Siran and W.D. Wallis, “Decomposing cubes," *J. Austral. Math. Soc.*, ser. A, vol. 61, pp. 119-128, 1996.

[14] N. Karisani and E.S. Mahmoodian, “On the construction of tree decompositions of hypercubes," *Bulletin of the Institute of Combinatorics and its Applications*, vol. 72, pp. 76 - 82, 2014.

[15] M. Kobeissia and M. Mollard, “Disjoint cycles and spanning graphs of hypercubes," *Discrete Math.*, vol. 288, pp. 73-87, 2004.

[16] S. C. Ku, B. F. Wang and T. K. Hung, “Constructing edge disjoint spanning trees in product networks," *IEEE Trans. Parallel and Distrib. Syst.*, vol. 14, no. 3, pp. 213-221, 2003.

[17] J. Laborde and S Hebbare, “Another characterization of hypercubes," *Discrete Math.*, vol. 39, pp. 161-166, 1982.

[18] F. Leighton, “Introduction to Parallel Algorithms and Architectures: Arrays, Trees, Hypercubes," M. Kaufmann Publishers, San Mateo, California, 1992.

[19] M. Mollard and M. Ramras, “Edge decompositions of hypercubes by paths and by cycles," *Graphs Combin.*, vol.31, no. 3, pp. 729-741, 2015.

[20] M. Ramras, “Fundamental subsets of edges of hypercubes," *Ars Combin.*, vol. 46, pp. 3-24, 1997.

[21] M. Ramras, “Symmetric edge-decompositions of hypercubes," *Graphs Combin.*, vol. 7, no. 1, pp. 65-87, 1991.

[22] J. Roskind and R.E. Tarjan, “A note on finding minimum-cost edge-disjoint spanning trees," *Math. of Operations Research*, vol. 10, no. 4, pp. 701-708, 1985.

[23] S.W. Song, “Towards a simple construction method for Hamiltonian decomposition of the hypercube," *Discrete Math. Theoret. Comput. Sci.*, ser. DIMACS, vol. 21, pp. 297-306, 1995.

[24] S. Wagner and M. Wild, “Decompositions of hypercubes Q_n into n isomorphic edge-disjoint trees," *Discrete Math.*, vol. 312, no. 10, pp. 1819 - 1822, 2012.

[25] D. B. West, *Introduction To Graph Theory*, 2nd ed. New Jersey: Pearson Education Inc., 2001.

Paper Presentation

A paper entitled “Decomposition of Hypercubes" was presented in the International Conference on Mathematics, organized by the Department of Mathematics, Providence College Coonoor, Tamil Nadu, India, on 04-08-2017.

Paper Publication

A. Roy and J. V. Kureethara, “Decompostion of Hypercubes," *Int. J. Pure and Appl. Math.*, to be published.